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Approximation of Powers of x by Polynomials

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1. INTRODUCTION

Bell and Shah have used oscillating generalized polynomials [2–4] to find the best uniformly approximating polynomial of degree n on [0, 1] to functions of the form $f(x) = x^r$, where r is a positive rational number. They then determined lower bounds for

$$E_n(r) = \min_{c_i} \max_{0 \leqslant x \leqslant 1} | x^r - (c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n)|.$$

This work was motivated by Bernstein's results [5] on the approximation of $|x|$ on $[-1, 1]$, which is equivalent to having $r = \frac{1}{2}$ and approximating on [0, 1].

In this paper we study the functions

$$E_n(\alpha) = \min_{c_i} \max_{0 \leqslant x \leqslant 1} | x^\alpha - (c_0 + c_1 x + \cdots + c_n x^n)|,$$

$$E'_n(\alpha) = \min_{c_i} \max_{0 \leqslant x \leqslant 1} | x^\alpha - (c_1 x + c_2 x^2 + \cdots + c_n x^n)|,$$

where $n \in N$, $\alpha > 0$ and c_i is real for each i. In so doing, the properties of Chebychev polynomials and of oscillating generalized polynomials are extremely useful in finding upper and lower bounds for $E_n(\alpha)$ for some α's. In particular, Lemma 1 enables us to find greater lower bounds of $E_n(p/q)$ for certain positive integers p and q than were previously known. Similarly, smaller upper bounds for $E_n(\alpha)$ are also found when $1 < \alpha < n$ and $\alpha \notin N$. The theory becomes much more complete when we show that each of $E_n(\alpha)$ and $E'_n(\alpha)$ is strictly monotonic in certain intervals.

2. OSCILLATING GENERALIZED POLYNOMIALS

Let $0 \leqslant \alpha(0) < \alpha(1) < \cdots < \alpha(n)$ be a given set of rational numbers. Then $p(x) = c_0 x^{\alpha(0)} + c_1 x^{\alpha(1)} + \cdots + c_n x^{\alpha(n)}$, where c_i are real is said to

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be a generalized polynomial (g.p.). If $\max_{0\leqslant x\leqslant 1} |p(x)|$ is attained for exactly $n+1$ values of x in [0, 1], then $p(x)$ is said to be an oscillating generalized polynomial (o.g.p.) in [0, 1] with exponents $\alpha(0), \alpha(1),..., \alpha(n)$. (We write for notational convenience $\alpha(j)$ for α_j.)

The following facts about g.p.'s and o.g.p.'s are stated: (i)–(vi) [2] and (vii) [6, p. 29]).

(i) (Property $\mathscr{D}$) (A) For every set of nonzero real numbers $\{c_0, c_1,..., c_n\}$ and every set of rational numbers $\{\alpha(0), \alpha(1),..., \alpha(n)\}$ with $0 \leqslant \alpha(0) < \alpha(1) < \cdots < \alpha(n)$, the number of zeros, a zero of order k counted as k zeros, in (0, 1] of the generalized polynomial

$$p(x) = c_0 x^{\alpha(0)} + c_1 x^{\alpha(1)} + \cdots + c_2 x^{\alpha(2)} + \cdots + c_n x^{\alpha(n)}$$

is at most equal to the number of variations of sign in the sequence $\{c_0, c_1,..., c_n\}$.

(B) With the sets $\{c_0, c_1,..., c_n\}$ and $\{\alpha(0), \alpha(1),..., \alpha(n)\}$ as in (A), the number of zeros, a zero of order k counted as k zeros, in (0, 1] of $p'(x)$ is at most equal to the number of variations of sign in the sequence $\{c_0, c_1,..., c_n\}$.

(ii) To a given finite set of nonnegative exponents, there corresponds an o.g.p. in [0, 1] which is unique except for a constant factor.

(iii) Write $M = \max_{0\leqslant x\leqslant 1} |p(x)|$. An o.g.p. $p(x)$ assumes the values $\pm M$ alternately at $n+1$ points in [0, 1].

(iv) Let $p(x) = \sum_{j=0}^{n} A_j x^{\alpha(j)}$ be an o.g.p. in [0, 1] and let $q(x) = \sum_{j=0}^{n} B_j x^{\alpha(j)}$ (all B_j real) be another generalized polynomial. Suppose $B_j = A_j$ for at least one j where $\alpha(j) > 0$. Then $\max_{0\leqslant x\leqslant 1} |q(x)| > \max_{0\leqslant x\leqslant 1} |p(x)|$.

(v) Let $p(x) = a_0 x^{\alpha(0)} + \sum_{k=1}^{n} a_k x^{\alpha(k)}$ and $q(x) = a_0 x^{\alpha(0)} + \sum_{k=1}^{n} b_k x^{\beta(k)}$ be o.g.p.'s such that $0 < \alpha(0) < \alpha(1) < \beta(1) < \cdots < \alpha(n) < \beta(n)$. Then $\max_{0\leqslant x\leqslant 1} |p(x)| < \max_{0\leqslant x\leqslant 1} |q(x)|$.

(vi) The coefficients of an o.g.p. $p(x) = a_0 x^{\alpha(0)} + a_1 x^{\alpha(1)} + \cdots + a_n x^{\alpha(n)}$ alternate in sign.

(vii) $E_n(\alpha) > E'_n(\alpha)/2$ for $\alpha > 0$ and rational.

3. Application of Oscillating Generalized Polynomials

Lemma 1. *Let*

$$p(x) = x^{\alpha(0)} + a_1 x^{\alpha(1)} + a_2 x^{\alpha(2)} + \cdots + a_n x^{\alpha(n)}$$

and

$$q(x) = x^{\alpha(0)} + b_1 x^{\beta(1)} + b_2 x^{\beta(2)} + \cdots + b_n x^{\beta(n)}$$

be the unique o.g.p.'s with 1 *as the coefficient of* $x^{\alpha(0)}$ *and positive rational exponents* $\{\alpha(0), \alpha(1), \alpha(2),..., \alpha(n)\}$ *and* $\{\alpha(0), \beta(1), \beta(2),..., \beta(n)\}$, *respectively, where* $0 < \alpha(0) < \alpha(1) < \cdots < \alpha(n)$ *and* $0 < \alpha(0) < \beta(1) < \cdots < \beta(n)$ *and for* $i = 1, 2,..., n$, $\alpha(i) < \beta(i)$. *Then* $\max_{0\leqslant x\leqslant 1} |p(x)| < \max_{0\leqslant x\leqslant 1} |q(x)|$.

Proof. The α's and β's in this argument are all to be rational. First choose $\{\beta(1, 1), \beta(2, 1),..., \beta(n, 1)\}$ by $\alpha(1) < \beta(1, 1) < \min\{\alpha(2), \beta(1)\}$ and for $i = 2, 3,..., n$, let $\beta(i, 1) \in (\max\{\alpha(i), \beta(i-1)\}, \beta(i))$. Next suppose for $j \in N$ with $1 < j < n - 1$ that $\{\beta(1,j), \beta(2,j),..., \beta(n,j)\}$ has been chosen so that $\alpha(1) < \beta(1,j) < \beta(1,j-1) < \alpha(2) < \beta(2,j) < \beta(2,j-1) < \alpha(3) < \cdots < \alpha(j) < \beta(j,j) < \min\{\alpha(j+1), \beta(j,j-1)\}$ with $\beta(i,j) \in (\max\{\alpha(i), \beta(i-1,j-1)\}, \beta(i,j-1))$ for $i = j+1, j+2,..., n$. Then choose $\{\beta(1,j+1), \beta(2,j+1),..., \beta(n,j+1)\}$ so that $\alpha(1) < \beta(1,j+1) < \beta(1,j) < \alpha(2) < \beta(2,j+1) < \beta(2,j) < \alpha(3) < \beta(3,j+1) < \beta(3,j) < \cdots < \alpha(j) < \beta(j,j+1) < \beta(j,j) < \alpha(j+1) < \beta(j+1,j+1) < \min\{\beta(j+1,j), \alpha(j+2)\}$ and let $\beta(i,j+1) \in (\max\{\beta(i-1,j), \alpha(i)\}, \beta(i,j))$ for $i = j+2, j+3,..., n$. Now for each $i = 1, 2,..., n-1$, define

$$p_i(x) = x^{\alpha(0)} + b_1^{(i)} x^{\beta(1,i)} + b_2^{(i)} x^{\beta(2,i)} + \cdots + b_n^{(i)} x^{\beta(n,i)}$$

to be the unique o.g.p. with exponents $\{\alpha(0), \beta(1, i), \beta(2, i),..., \beta(n, i)\}$ and 1 as the coefficient of $x^{\alpha(0)}$. Then by (v) of Section 2,

$$\max_{0\leqslant x\leqslant 1} |p(x)| < \max_{0\leqslant x\leqslant 1} |p_{n-1}(x)| < \max_{0\leqslant x\leqslant 1} |p_{n-2}(x)|$$
$$< \cdots < \max_{0\leqslant x\leqslant 1} |p_1(x)| < \max_{0\leqslant x\leqslant 1} |q(x)|.$$

PROPOSITION 2. *Let* $n, k \in N$ *with* $k \geqslant 4$. *Then* $E_n(1/k) > 1/2(2n+1)$.

Proof. Let $x^{(1/k)} + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be the unique o.g.p. with exponents $\{1/k, 1, 2,..., n\}$ and with 1 as the coefficient of $x^{(1/k)}$. Then

$$E_n(1/k) > \tfrac{1}{2}E'_n(1/k) = \tfrac{1}{2} \max_{0\leqslant x\leqslant 1} | x^{1/k} + a_1 x + a_2 x^2 + \cdots + a_n x^n |$$
$$= \tfrac{1}{2} \max_{0\leqslant x\leqslant 1} | x + a_1 x^k + a_2 x^{2k} + \cdots + a_n x^{nk} |, \quad (1)$$

by (vii) of Section 2. Also by Theorem 1, it follows that

$$\max_{0\leqslant x\leqslant 1} | x + a_1 x^k + a_2 x^{2k} + \cdots + a_n x^{nk} |$$
$$> \max_{0\leqslant x\leqslant 1} | T_{2n+1}(x)/(2n+1)| = \frac{1}{2n+1}, \quad (2)$$

where $T_{2n+1}(x)$ is the Chebychev polynomial of degree $2n+1$. By (1) and (2) it follows that $E_n(1/k) > 1/2(2n+1)$.

PROPOSITION 3. *Let $p(x) = x + a_1x^3 + a_2x^6 + \cdots + a_nx^{3n}$ be the unique o.p. with exponents $\{1, 3, 6,..., 3n\}$ and with 1 as the coefficient of x. Then $E'_n(1/3) = \max_{0\leqslant x\leqslant 1} |p(x)| \geqslant 1/3(2n-1)$ with equality if and only if $n = 1$.*

Proof. Let $n \geqslant 2$ and $r(x) = x + c_2x^6 + c_3x^9 + \cdots + c_nx^{3n}$ be the unique oscillating polynomial (o.p.) with exponents $\{1, 6, 9,..., 3n\}$ and with 1 as the coefficient of x. Also, the unique o.p. with exponents $\{1, 3, 5,..., 2n-1\}$ and with 1 as the coefficient of x is $T_{(2n-1)}(x)/(2n-1)$. Since $1 = 1, 3 < 6, 5 < 9,..., 2n-1 < 3n$, it follows by Theorem 1 that

$$\max_{0\leqslant x\leqslant 1} |r(x)| > \max_{0\leqslant x\leqslant 1} |T_{(2n-1)}(x)/(2n-1)| = 1/(2n-1).$$

Now, if the technique used in [4, p. 273; 5, pp. 9, 10] is used with the fact that $\max_{0\leqslant x\leqslant 1} |r(x)| > 1/(2n-1)$ and the transformation $y = x^{1/3}$, it is immediate that $E'_n(1/3) = \max_{0\leqslant x\leqslant 1} |p(x)| > 1/3(2n-1)$.

If $n = 1$, then $p(x) = -T_3(x)/3$ and $\max_{0\leqslant x\leqslant 1} |p(x)| = 1/3(2n-1)$.

COROLLARY 4. $E_n(1/3) = \min_{c_i} \max_{0\leqslant x\leqslant 1} | x^{1/3} - (c_0 + c_1x + \cdots + c_nx^n)| > 1/6(2n-1)$.

Proof. This follows by Proposition 3 and (vii) of Section 2.

PROPOSITION 5. (a) *If $p, q \in N$ with $3p < q$, then $E_n(p/q) > 1/2(2n+1)$.*

(b) *If $p, q \in N$ with $2p < q$, then $E_n(p/q) > 1/4(1 + 2^{1/2})(2n-1)$.*

Proof. (a) Let $r(x) = x^{p/q} + b_1x + b_2x^2 + \cdots + b_nx^n$ be the unique o.g.p. with exponents $\{p/q, 1, 2,..., n\}$ and 1 as the coefficent of $x^{p/q}$. Let $\tilde{r}(x) = x + b_1x^{(q/p)} + b_2x^{2(q/p)} + \cdots + b_nx^{n(q/p)}$. Then for $i = 2, 3,..., n$, $(i)(q/p) - (i-1)(p/q) = p/q > 3$. Therefore by Theorem 1,

$$E_n(p/q) = \max_{0\leqslant x\leqslant 1} |r(x)| = \max_{0\leqslant x\leqslant 1} |\tilde{r}(x)|$$
$$> \max_{0\leqslant x\leqslant 1} |T_{(2n+1)}(x)/(2n+1)| = 1/(2n+1).$$

Consequently, $E_n(p/q) > 1/2(2n+1)$ by (vii) of Section 2.

(b) Let $r(x)$ and $\tilde{r}(x)$ be as in part (a). Define $t(x) = x + c_1x^2 + c_2x^4 + \cdots + c_nx^{2n}$ to be the unique o.p. with exponents $\{1, 2, 4,..., 2n\}$ and with 1 as the coefficient of x. By Theorem 1,

$$\max_{0\leqslant x\leqslant 1} |r(x)| = \max_{0\leqslant x\leqslant 1} |\tilde{r}(x)| > \max_{0\leqslant x\leqslant 1} |t(x)|.$$

By [6, pp. 27, 28], $\max_{0\leqslant x\leqslant 1} |t(x)| \geqslant 1/2(1 + 2^{1/2})(2n - 1)$. If $n \geqslant 2$, $E_n(p/q) > 1/4(1 + 2^{1/2})(2n - 1)$. For $n = 1$, let $p(x) = x + a_1x^2$, $s(x) = x^{p/q} + b_1x$, and $\tilde{s}(x) = x + b_1x^{q/p}$ be the unique o.g.p.'s. Then $p(x) = x - (1/2 + 1/(2^{1/2}))\, x^2$ by [5, p. 28] and

$$\max_{0\leqslant x\leqslant 1} | s(x)| = \max_{0\leqslant x\leqslant 1} | \tilde{s}(x)| > \max_{0\leqslant x\leqslant 1} | p(x)| = \frac{1}{2(1 + 2^{1/2})(2n - 1)}$$

by Theorem 1. Therefore, by (vii) of Section 2, $E_n(p/q) > 1/4(1 + 2^{1/2}) \times 1/(2n - 1)$.

LEMMA 6. *If* $\alpha > 0$ *and* $\alpha \notin N$, *then* $E_n(\alpha), E'_n(\alpha) > 0$.

This is obvious.

PROPOSITION 7. *Let* α *be so that* $1 < \alpha < n$. *Then* $E_n(\alpha) < 1/\{2(n - [\alpha - 1]) + 1\}$, *where* $[\alpha - 1]$ *is the greatest integer* $\leqslant \alpha - 1$.

Proof. If $[\alpha] = \alpha$, then by Theorem 6, the conclusion is trivial since $E_n(\alpha) = 0$. Therefore suppose that $[\alpha] \neq \alpha$. Next let

$$x^\alpha + b_2x^{([\alpha]+1)} + b_3x^{([\alpha]+2)} + \cdots + b_{j-1}x^{n-1} + b_jx^n,$$

with $j = n - [\alpha - 1]$, be the unique o.g.p. with exponents $\{\alpha, [\alpha] + 1, [\alpha] + 2,..., n - 1, n\}$ and with 1 as the coefficient of x^α. It then follows by the Alternation Theorem and by the definition of o.g.p.'s that

$$E_n(\alpha) < \max_{0\leqslant x\leqslant 1} | x^\alpha + b_2x^{([\alpha]+1)} + b_3x^{([\alpha]+2)} + \cdots + b_jx^n |.$$

Also let

$$x^\alpha + c_2x^{3\alpha} + c_3x^{5\alpha} + \cdots + c_jx^{\{2(n-[\alpha-1])-1\}\alpha}$$

be the unique o.g.p. with exponents $\{\alpha, 3\alpha, 5\alpha,..., (2(n - [\alpha - 1]) - 1)\,\alpha\}$ and with 1 as the coefficient of x^α. Then by Theorem 1,

$$\max_{0\leqslant x\leqslant 1} | x^\alpha + b_2x^{([\alpha]+1)} + b_3x^{([\alpha]+2)} + \cdots + b_jx^n |$$

$$< \max_{0\leqslant x\leqslant 1} | x^\alpha + c_2x^{3\alpha} + c_3x^{5\alpha} + \cdots + c_jx^{\{2(n-[\alpha-1])-1\}\alpha} |$$

$$= \frac{1}{\{2(n - [\alpha - 1]) - 1\}},$$

since $[\alpha] + 1 < 3\alpha$, $[\alpha] + 2 < 5\alpha,..., n < \{2(n - [\alpha - 1]) - 1\}\, \alpha$ and $x^\alpha + c_2x^{3\alpha} + c_3x^{5\alpha} + \cdots + c_jx^{\{2(n-[\alpha-1])-1\}\alpha} = T_{\{2(n-[\alpha-1])-1\}}(x^\alpha)/\{2(n - [\alpha - 1]) - 1\}$.

4. Monotonicity and Continuity of E_n and E_n'

First, it is rather routine to show the following.

PROPOSITION 8. *Each of E_n and E'_n is a continuous function on* $(0, \infty)$.

COROLLARY 9. $E_n(1/3) \geqslant 1/2(n+1)$ (*Compare this with Corollary* 4.)

Proof. This follows by the continuity of E_n and by Proposition 5.

Now let each of $\alpha(1), \alpha(2),..., \alpha(n), \alpha(n+1), \beta(1), \beta(2),..., \beta(n)$, and $\beta(n+1)$ be a rational number with $0 < \alpha(1) < \alpha(2) < \cdots < \alpha(n) < \alpha(n+1)$ and $0 < \beta(1) < \beta(2) < \cdots < \beta(n) < \beta(n+1)$ and suppose a_0 and b_0 are nonzero with the same sign. Let each of

$$p(x) = a_0 + a_1x^{\alpha(1)} + a_2x^{\alpha(2)} + \cdots + a_nx^{\alpha(n)} + a_{n+1}x^{\alpha(n+1)}$$

and

$$q(x) = b_0 + b_1x^{\beta(1)} + b_2x^{\beta(2)} + \cdots + b_nx^{\beta(n)} + b_{n+1}x^{\beta(n+1)}$$

be an o.g.p. If $\max_{0\leqslant x\leqslant 1} |p(x)| = \max_{0\leqslant x\leqslant 1} |q(x)| = M > 0$, then $p(x)$ and $q(x)$ are said to be an M–n-oscillating pair which is denoted by $\langle p(x), q(x)\rangle$.

LEMMA 10. *Let $\langle p(x), q(x)\rangle$ be an M–n-oscillating pair and let $0 = p_0 < p_1 < \cdots < p_n < p_{n+1} = 1$ and $0 = q_0 < q_1 < \cdots < q_n < q_{n+1} = 1$ be the points in $[0, 1]$ at which $p(x)$ and $q(x)$ take on their extreme values, respectively. Then there are two zeros of $(p - q)(x)$ in $(0, \max\{p_1, q_1\}]$ if $(p - q)(\max\{p_1, q_1\}) = 0$ and there is one zero in $(0, \max\{p_1, q_1\}]$ if $(p - q)(\max\{p_1, q_1\}) \neq 0$.*

Proof. Suppose $(p - q)(\max\{p_1, q_1\}) \neq 0$. Then $p_1 \neq q_1$. Suppose $p_1 < q_1$. Then either

(i) $(p - q)(p_1) > 0$ and $(p - q)(q_1) \leqslant 0$ or
(ii) $(p - q)(p_1) < 0$ and $(p - q)(q_1) \geqslant 0$.

Consequently there is a zero of $(p - q)(x)$ in $(0, \max\{p_1, q_1\}]$ since $(p - q)(x)$ is continuous.

Suppose $(p - q)(\max\{p_1, q_1\}) = 0$ and $p_1 \leqslant q_1$. Then $(p - q)(q_1) = 0$ means that $p(q_1) = q(q_1) = M$ and $p'(q_1) = q'(q_1) = 0$. Consequently $(p - q)'(q_1) = 0$ and q_1 is a double zero of $(p - q)(x)$ by Ahlfors [1, pp. 126, 127]. Then $(p - q)(x)$ has two zeros in $(0, \max\{p_1, q_1\}]$.

PROPOSITION 11. *If $\langle p(x), q(x)\rangle$ is an M–n-oscillating pair, then $(p - q)(x)$ has $n + 1$ zeros in* $(0, 1]$.

Proof. The notation is as in Lemma 10 and the proof proceeds by mathematical induction. Let $S = \{j \in N$: for each M–n-oscillating pair $\langle p(x), q(x)\rangle$ with $n \geqslant j$, there are j zeros in $(0, \max\{p_j, q_j\}]$ if $(p - q)(\max\{p_j, q_j\}) \neq 0$ and there are $j + 1$ zeros in $(0, \max\{p_j, q_j\}]$ if $(p - q)(\max\{p_j, q_j\}) = 0\}$. By Lemma 10, $1 \in S$.

Suppose $j \in S$. Let $n \geqslant j + 1$ and let $\langle p(x), q(x)\rangle$ be an M–n-oscillating pair. Then either

(A) $(p - q)(\max\{p_j, q_j\}) \neq 0$ or

(B) $(p - q)(\max\{p_j, q_j\}) = 0$.

Case A

Suppose $(p - q)(\max\{p_j, q_j\}) \neq 0$. Then there are j zeros of $(p - q)(x)$ in $(0, \max\{p_j, q_j\}]$. There are two possibilities.

Subcase A1. $(p - q)(\max\{p_{j+1}, q_{j+1}\}) = 0$ implies that there are $j + 2$ zeros in $(0, \max\{p_{j+1}, q_{j+1}\}]$ because of a double zero at $\max\{p_{j+1}, q_{j+1}\}$.

Subcase A2. Suppose $(p - q)(\max\{p_{j+1}, q_{j+1}\}) \neq 0$.

(i) If $p_j < q_j$ and $p_{j+1} < q_{j+1}$, then by A and A2 it follows that $(p - q)(q_j) < 0$ with $(p - q)(q_{j+1}) > 0$ or $(p - q)(q_j) > 0$ with $(p - q)(q_{j+1}) < 0$.

(ii) If $q_j < p_j$ and $p_{j+1} < q_{j+1}$, then $(p - q)(p_j) < 0$ with $(p - q)(p_{j+1}) > 0$ or $(p - q)(p_j) > 0$ with $(p - q)(p_{j+1}) < 0$.

(iii) If $p_j < q_j$ and $q_{j+1} < p_{j+1}$, then $(p - q)(q_j) < 0$ with $(p - q)(q_{j+1}) > 0$ or $(p - q)(q_j) > 0$ with $(p - q)(q_{j+1}) < 0$.

(iv) If $q_j < p_j$ and $q_{j+1} < p_{j+1}$, then by A and A2, $(p - q)(p_j) < 0$ with $(p - q)(p_{j+1}) > 0$ or $(p - q)(p_j) > 0$ with $(p - q)(p_{j+1}) < 0$.

By (i)–(iv) it is clear that there is a zero of $(p - q)(x)$ in $(\max\{p_j, q_j\}, \max\{p_{j+1}, q_{j+1}\}]$ and $j + 1$ zeros in $(0, \max\{p_{j+1}, q_{j+1}\}]$.

Case B

Suppose $(p - q)(\max\{p_j, q_j\}) = 0$. Therefore there are $j + 1$ zeros of $(p - q)(x)$ in $(0, \max\{p_j, q_j\}]$. Then there are two possibilities.

Subcase B1. Suppose $(p - q)(\max\{p_{j+1}, q_{j+1}\}) = 0$. Then $(p - q)(x)$ has a double zero at $\max\{p_{j+1}, q_{j+1}\}$ and has $j + 3$ zeros in $(0, \max\{p_{j+1}, q_{j+1}\}]$.

Subcase B2. Suppose $(p - q)(\max\{p_{j+1}, q_{j+1}\}) \neq 0$. Then there are $j + 1$ zeros in $(0, \max\{p_{j+1}, q_{j+1}\}]$ since there are $j + 1$ zeros in $(0, \max\{p_j, q_j\}]$.

Consequently, $j + 1 \in S$ and $S = N$. It follows that for an M–n-oscillating pair $\langle p(x), q(x) \rangle$, $(p - q)(x)$ has n zeros in $(0, \max\{p_n, q_n\}]$. Since $p_{n+1} = q_{n+1} = 1$, $(p - q)(1) = 0$ and $(p - q)(x)$ has $n + 1$ zeros in $(0, 1]$.

LEMMA 12. *Let r be a positive rational number. If s is also a rational number with $s \in (r, r + r/n)$, then $1/s < 1/r < 2/s < 2/r < \cdots < n/s < n/r$.*

Proof. Let $s \in (r, r + r/n)$. Then $s < r + r/n$ or $n/r < (n + 1)/s$. Suppose for some $i \in N$ with $i < n$ that $i/r \geqslant (i + 1)/s$. Therefore $is \geqslant (i + 1) r$ and $(n - i) s + is \geqslant (n - i) r + (i + 1) r$, and $n/r \geqslant (n + 1)/s$. This is a contradiction.

PROPOSITION 13. *Let r and s be rational numbers with $r \in (0, 1)$ and $s \in (r, \min\{r + r/n, 1\})$, then $E_n(r) \neq E_n(s)$.*

Proof. Let $p_r(x) = b_0 + x^r + b_1x + b_2x^2 + \cdots + b_nx^n$ and $p_s(x) = c_0 + x^s + c_1x + c_2x^2 + \cdots + c_nx^n$ be the unique o.g.p.'s with exponents $\{0, r, 1, 2, \ldots, n\}$ and $\{0, s, 1, 2, \ldots, n\}$, respectively, and with 1 as the coefficient of each of x^r and x^s. Then each of

$$\tilde{p}_r(x) = b_0 + x + b_1x^{1/r} + \cdots + b_nx^{n/r}$$

and

$$\tilde{p}_s(x) = c_0 + x + c_1x^{1/s} + \cdots + c_nx^{n/s}$$

is also an o.g.p. Suppose $E_n(s) = E_n(r) = M$. Then $\langle \tilde{p}_r(x), \tilde{p}_s(x) \rangle$ is an M–n-oscillating pair. Clearly, $b_0 = -E_n(r) = -E_n(s) = c_0$, since the coefficients of $\tilde{p}_r(x)$ and $\tilde{p}_s(x)$ alternate in sign and o.g.p.'s take on extreme values at both 0 and 1.

$(\tilde{p}_s - \tilde{p}_r)(x) = c_1x^{1/s} - b_1x^{1/r} + c_2x^{2/s} - b_2x^{2/r} + \cdots + c_nx^{n/s} - b_nx^{n/r}$ and by Lemma 12, $1/s < 1/r < 2/s < 2/r < \cdots < n/s < n/r$. By property $\mathscr{D}$, $(\tilde{p}_s - \tilde{p}_r)(x)$ has at most n zeros in $(0, 1]$. However, by Proposition 11, $(\tilde{p}_s - \tilde{p}_r)(x)$ has $n + 1$ zeros in $(0, 1]$. This contradiction implies that $E_n(r) \neq E_n(s)$.

PROPOSITION 14. *Let $r \in (0, 1)$ and $s \in (0, 1] \cap (r, r + r/n)$ be rational numbers. Then $E_n(r) > E_n(s)$.*

Proof. First let $s < 1$. Suppose $E_n(r) < E_n(s)$. Let $\tilde{p}_r(x) = b_0 + x + b_1x^{1/r} + \cdots + b_nx^{n/r}$ and $\tilde{p}_s(x) = c_0 + x + c_1x^{1/s} + \cdots + c_nx^{n/s}$ be as in Proposition 13. Then by (vi) of Section 2, $E_n(r) = -b_0$ and $E_n(s) = -c_0$. Consequently

$$(\tilde{p}_s - \tilde{p}_r)(x) = (c_0 - b_0) + c_1 x^{1/s} - b_1 x^{1/r} + c_2 x^{2/s} - b_2 x^{2/r} + \cdots + c_n x^{n/s} - b_n x^{n/r}$$

has only n sign changes in the finite sequence $\{(c_0 - b_0), c_1, -b_1, c_2, -b_2, \ldots, c_n, -b_n\}$, since $-b_0 = E_n(r) < E_n(s) = -c_0$ and $(c_0 - b_0) < 0$. Therefore $(\tilde{p}_s - \tilde{p}_r)(x)$ has at most n zeros in $(0, 1]$, by property $\mathscr{D}$.

On the other hand let $0 = p_0 < p_1 < \cdots < p_n < p_{n+1} = 1$ be the $n + 2$ points of the interval $[0, 1]$ at which $\tilde{p}_s(x)$ takes on its extreme values in an alternating fashion.

Since $\tilde{p}_s(x)$ is continuous on $[0, 1]$, for each $i = 1, 2, \ldots, n + 1$, the range of $\tilde{p}_s(x)$ on $[p_{i-1}, p_i]$ is $[c_0, -c_0]$. Therefore it is easily shown that there exists $z_i \in (p_{i-1}, p_i)$ with $(\tilde{p}_s - \tilde{p}_r)(z_i) = 0$ and $(\tilde{p}_s - \tilde{p}_r)(x)$ has at least $n + 1$ zeros in $(0, 1]$. This is a contradiction and $E_n(r) > E_n(s)$, since by Proposition 13, $E_n(r) \neq E_n(s)$.

If $s = 1$, by Lemma 6, $E_n(r) > 0$ and $E_n(s) = 0$ and $E_n(r) > E_n(s)$.

THEOREM 15. (a) *E_n is strictly decreasing on* $(0, 1]$, *and* (b) *E_n is strictly increasing on* $[n, \infty)$.

Proof. (a) Since E_n is continuous, the result follows if E_n is strictly decreasing on the rational numbers in $(0, 1]$. Let r and s be rational numbers with $0 < r < s \leqslant 1$. Then there exists a smallest positive integer j so that $r + j(r/2n) > s$. For each $k = 1, 2, \ldots, j$, let $r_k = r + k(r/2n)$. Now $r = r_0 < r_1 < r_2 < \cdots < r_{j-1} < r_j$ with $r_{j-1} \leqslant s < r_j$ and $r_k < r_{k-1} + r_{(k-1)}/n$. Consequently, by Proposition 14, $E_n(r_k) < E_n(r_{k-1})$ for $k = 1, 2, \ldots, j - 1$, and either $E_n(r_{j-1}) \geqslant E_n(s)$ or $E_n(r_{j-1}) > E_n(s)$. Therefore $E_n(r) > E_n(s)$.

(b) As in part (a), it is only necessary to show that E_n is strictly increasing on the rational numbers in $[n, \infty)$. In the following, r and s are rational numbers. First suppose $n < r < s$. For if $n = r$, then by Lemma 6, the proof is trivial. Let

$$p_r(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + x^r$$

and

$$p_s(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + x^s$$

be the unique o.g.p.'s with exponents $\{0, 1, 2, 3, \ldots, n, r\}$ and $\{0, 1, 2, 3, \ldots, n, s\}$, respectively, and with 1 as the coefficient of each of x^r and x^s. Therefore, by (vi) of Section 2, for each $i = 0, 1, 2, \ldots, n$, b_i and c_i have the same sign. Let

$$\tilde{p}_r(x) = b_0 + b_1 x^{1/r} + \cdots + b_n x^{n/r} + x$$

and

$$\tilde{p}_s(x) = c_0 + c_1 x^{1/s} + \cdots + c_n x^{n/s} + x.$$

Each of $\tilde{p}_r(x)$ and $\tilde{p}_s(x)$ is also an o.g.p.

Case A. Suppose n is even. Then both b_0 and c_0 are negative with $E_n(r) = -b_0$ and $E_n(s) = -c_0$.

(i) Suppose $E_n(s) = E_n(r)$. Then $c_0 = b_0$. Consequently,

$$(p_s - p_r)(x) = c_1 x^{1/s} - b_1 x^{1/r} + c_2 x^{2/s} - b_2 x^{2/r} + \cdots + c_n x^{n/s} - b_n x^{n/r}$$

By Lemma 12, $1/s < 1/r < \cdots < n/s < n/r$; and property $\mathscr{D}$ shows by examination of $\{c_1, -b_1, c_2, -b_2, \ldots, c_n, -b_n\}$, that $(\tilde{p}_s - \tilde{p}_r)(x)$ has at most n zeros in $(0, 1]$. This is a contradiction since $\langle \tilde{p}_r(x), \tilde{p}_s(x) \rangle$ is an M–n-oscillating pair and $E_n(r) \neq E_n(s)$.

(ii) Suppose that $E_n(s) < E_n(r)$. Then $-c_0 = E_n(s) < E_n(r) = -b_0$ and $c_0 - b_0 > 0$. By (vi) of Section 2, $c_1 > 0$ and the sequence $\{(c_0 - b_0), c_1, -b_1, c_2, -b_2, \ldots, c_n, -b_n\}$ of coefficients of

$$(\tilde{p}_s - \tilde{p}_r)(x) = (c_0 - b_0) + c_1 x^{1/s} - b_1 x^{1/r} + c_2 x^{2/s} - b_2 x^{2/r} + \cdots + c_n x^{n/s} - b_n x^{n/r}$$

has n sign changes. Since $0 < 1/s < 1/r < \cdots < n/r < n/s$, by property $\mathscr{D}$, $(\tilde{p}_s - \tilde{p}_r)(x)$ has at most n zeros in $(0, 1]$. However, as in the proof of Proposition 14, it is clear that $(\tilde{p}_r - \tilde{p}_s)(x)$ has $n + 1$ zeros in $(0, 1)$. Therefor $E_n(s) > E_n(r)$.

Case B. Suppose n is odd. This case is similar to Case A.

COROLLARY 16. *$E_n(\alpha) \leqslant 1/2^{(2n+1)}$ for $\alpha \in [n, n+1]$ and $E_n(\alpha) > 1/2^{(2n+1)}$ for $\alpha \in (n+1, \infty)$.*

Proof. Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + x^{n+1}$ be the unique o.p. with exponents $\{0, 1, 2, \ldots, n, n+1\}$ and with 1 as the coefficient of x^{n+1}. Then

$$E_n(n+1) = \max_{0 \leqslant x \leqslant 1} |p(x)|$$
$$= \max_{0 \leqslant x \leqslant 1} |a_0 + a_1 x^2 + a_2 x^4 + \cdots + a_n x^{2n} + x^{(2n+2)}|$$

and

$$\frac{T_{(2n+2)}(x)}{2^{(2n+1)}} = a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n} + x^{(2n+2)},$$

by uniqueness. By Theorem 15(b), $E_n(\alpha) \leqslant 1/[2^{(2n+1)}]$ for $\alpha \in [n, n+1]$ and $E_n(\alpha) > 1/[2^{(2n+1)}]$ for $\alpha \in (n+1, \infty)$ since $E_n(n+1) = 1/[2^{(2n+1)}]$.

THEOREM 17. (a) *E'_n is strictly decreasing on* $(0, 1]$ *and* (b) *E'_n is strictly increasing on* $[n, \infty)$.

Proof. (a) Since E'_n is continuous it is sufficient to show the monotonicity on the rational numbers. Also suppose $s \in (r, \min\{1, r + r/n\})$, since the technique of the proof of Theorem 15 can be used otherwise. Let

$$p_r(x) = x^r + b_1x + b_2x^2 + \cdots + b_nx^n$$
$$p_s(x) = x^s + c_1x + c_2x^2 + \cdots + c_nx^n$$

be the unique o.g.p.'s with exponents $\{r, 1, 2,..., n\}$ and $\{s, 1, 2,..., n\}$, respectively, and with 1 as the coefficient of each of x^r and x^s. Then

$$\tilde{p}_r(x) = x + b_1x^{1/r} + b_2x^{2/r} + \cdots + b_nx^{n/r}$$

and

$$\tilde{p}_s(x) = x + c_1x^{1/s} + c_2x^{2/s} + \cdots + c_nx^{n/s}$$

are also o.g.p.'s. Since $0 < 1 < 1/s < 1/r < 2/s < 2/r < \cdots < n/s < n/r$, by (v) of Section 2,

$$\max_{0\leqslant x\leqslant 1} |p_r(x)| = \max_{0\leqslant x\leqslant 1} |\tilde{p}_r(x)| > \max_{0\leqslant x\leqslant 1} |\tilde{p}_s(x)| = \max_{0\leqslant x\leqslant 1} |p_s(x)|$$

and $E'_n(r) > E'_n(s)$. If $s = 1$, it follows by Lemma 6 that $E'_n(r) > E'_n(s)$ since $E'_n(s) = 0$.

(b) This part is similar and is omitted.

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